

# Effect on normalized graph Laplacian spectrum by motif joining and duplication

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October 19, 2012

## Abstract

Here we investigate how the spectrum of the normalized graph Laplacian gets affected by certain graph operations like motif duplication and graph (or motif) joining.

## 1 Introduction

Let  $\Gamma = (V, E)$  be a connected and finite graph of order  $N$  with the vertex set  $V$  and the edge set  $E$ . If two vertices  $i, j \in V(\Gamma)$  are connected by an edge in  $E(\Gamma)$ , they are called neighbors,  $i \sim j$ . Let  $n_i$  be the degree of  $i \in V(\Gamma)$ , that is, the number of neighbors of  $i$ . For functions  $g : V \rightarrow \mathbb{R}$  we define the normalized graph Laplacian as

$$\Delta g(i) := g(i) - \frac{1}{n_i} \sum_{j, j \sim i} g(j). \quad (1)$$

Note that this operator (studied in [1, 3]) is different from the (algebraic) graph Laplacian operator,  $Lg(i) := n_i g(i) - \sum_{j: j \sim i} g(j)$  (see [5, 7, 9, 10, 4] for this operator). But our operator (1) is similar to the Laplacian:  $\mathcal{L}g(i) := g(i) - \sum_{j: j \sim i} \frac{1}{\sqrt{n_i n_j}} g(j)$  investigated in [6] and thus both have the same spectrum.

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<sup>\*</sup>This author thanks Dragan Stevanovic, Francesc Comellas and the other organizers for the nice meeting and good hospitality

Now we recall some of the basic properties of the eigenvalues and eigenfunctions of the operator (1) from [8, 5, 1, 3]. The normalized Laplacian is symmetric for the product

$$\langle g_1, g_2 \rangle := \sum_{i \in V} n_i g_1(i) g_2(i) \quad (2)$$

for real valued functions  $g_1, g_2$  on  $V(\Gamma)$ . Since  $(\Delta g, g) \geq 0$  all eigenvalues of  $\Delta$  are non-negative. The eigenvalue equation of  $\Delta$  is

$$\Delta f - \lambda f = 0 \quad (3)$$

where a nonzero solution  $f$  is called an eigenfunction corresponding to the eigenvalue  $\lambda$ . Let  $m_\lambda$  be the algebraic multiplicity of the eigenvalues  $\lambda$ . Now if we arrange all the eigenvalues in non-decreasing manner we have

$$\lambda_0 = 0 < \lambda_1 \leq \dots \leq \lambda_{n-1} \leq 2.$$

A graph is bipartite iff  $\lambda_{n-1} = 2$ . For a connected graph only the smallest eigenvalue  $\lambda_0$  is 0 with an constant eigenfunction. Hence, for all other eigenfunctions  $f$ , from (2) we get

$$\sum_{i \in V} n_i f(i) = 0 \quad (4)$$

The eigenvalue equation (3) become

$$\frac{1}{n_i} \sum_{j \sim i} f(j) = (1 - \lambda) f(i) \text{ for all } i. \quad (5)$$

In particular, when the eigenfunction  $f$  vanishes at  $i$ , then also  $\sum_{j \sim i} f(j) = 0$ , and conversely (for eigenvalue 1 converse is not always true).

## 2 Eigenvalue 1

For  $\lambda = 1$  the equation (5) becomes

$$\sum_{j \in \Gamma, j \sim i} f(j) = 0 \quad \forall i \in \Gamma. \quad (6)$$

Which is a special property of an eigenfunction for the eigenvalue 1 since it doesn't depend on the degrees of the vertices.

### Vertex Doubling

A vertex-doubling of a vertex  $p$  of  $\Gamma$  is to add a vertex  $q$  to  $\Gamma$  and connect it to all  $j$  in  $\Gamma$  whenever  $j \sim p$ . Vertex doubling of a vertex  $p$  of  $\Gamma$  ensures the eigenvalue 1, with an eigenfunction  $f_1$ , which takes value 1 at  $p$ , -1 at its double and 0 otherwise [1].

Now let  $\Gamma$  be a graph which does not have an eigenvalue 1. We want to produce a graph from  $\Gamma$  with eigenvalue 1 of multiplicity  $m$  by repeated-doubling of a vertex  $p$  of  $\Gamma$ . Now we double  $p$  repeatedly by adding one by one the vertices  $q_1, q_2, q_3, \dots, q_m$  and joining them to  $j \sim p$  and get the graphs  $\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \dots, \Gamma^{(m)}$ . We will show that  $\Gamma^{(i)}$  have the eigenvalue 1 with multiplicity  $i$ . By taking a set of eigenfunctions for  $\lambda = 1$  in  $\Gamma^{(i)}$  can be obtained as

$$f_j^{(i)}(x) = \begin{cases} 1 & \text{if } x = p, q_1, q_2, \dots, q_{j-1} \\ -j & \text{if } x = q_j \\ 0 & \text{else.} \end{cases} \quad (7)$$

for  $j=1,2,\dots,i$ .

## Motif Doubling

Let  $\Sigma$  be a connected subgraph of  $\Gamma$  with vertices  $p_1, \dots, p_m$ , containing all of  $\Gamma$ 's edges between those vertices, i.e. an induced subgraph. We call such a  $\Sigma$  a motif. Suppose  $\Sigma$  has an eigenvalue 1 with eigenfunction  $f_1^\Sigma$ . Then  $\Gamma^\Sigma$ , obtained by doubling  $\Sigma$  in  $\Gamma$ , ensures  $\lambda = 1$  with eigenfunction  $f_1^{\Gamma^\Sigma}$  such that,

$$f_1^{\Gamma^\Sigma}(p) = \begin{cases} f_1^\Sigma(p_\alpha) & \text{if } p = p_\alpha \in \Sigma \\ -f_1^\Sigma(p_\alpha) & \text{if } p = q_\alpha \\ 0 & \text{else.} \end{cases} \quad (8)$$

where  $q_\alpha$  denote the double of  $p_\alpha \in \Sigma$  [1].

Now we come to some different operation. We double  $\Sigma$  repeatedly for  $m$  times. Let  $\Sigma$  consists of the vertices  $p_1, p_2, \dots, p_k$ . At the first stage we get the graph  $\Gamma^{\Sigma^1}$  by doubling  $\Sigma$  once and let  $q_j^{(1)}$  in first duplicate of  $\Sigma$  corresponds to  $p_j$  in  $\Sigma$  for all  $j = 1, 2, \dots, k$  in  $\Sigma$ . Again at the second stage we get  $\Gamma^{\Sigma^2}$  with  $q_j^{(2)}$  in second duplicate of  $\Sigma$  corresponding to  $p_j$  in  $\Sigma$  for all  $j = 1, 2, \dots, k$ . Similarly at the  $m$ th stage we get  $\Gamma^{\Sigma^m}$  with the vertices  $q_j^{(m)}$  in  $m$ th duplicate of  $\Sigma$  corresponding to  $p_j$  in  $\Sigma$  for all  $j = 1, 2, \dots, k$ . We want to find a set of eigenfunctions for  $\lambda = 1$ . The following theorem gives a set of eigenfunctions for  $\lambda = 1$ .

**Theorem 2.1.**  $\Gamma^{\Sigma^i}, i = 1, 2, \dots, m$  has the eigenvalue 1 with a set of eigenfunction  $f_j^{\Gamma^{\Sigma^i}}, j = 1, 2, \dots, i$  all of which coincides with  $f_1^\Sigma$  on  $\Sigma$ .

*Proof.* For each  $j \in 1, 2, \dots, i$  we define

$$f_j^{\Gamma^{\Sigma^i}}(p) = \begin{cases} f_1^\Sigma(p_\alpha) & \text{if } p = p_\alpha \in \Sigma \\ f_1^\Sigma(p_\alpha) & \text{if } p = q_\alpha^{(l)}, j > 1, \quad 1 \leq l \leq j-1 \\ -j f_1^\Sigma(p_\alpha) & \text{if } p = q_\alpha^{(j)} \\ 0 & \text{elsewhere.} \end{cases} \quad (9)$$

for  $i = 1, 2, \dots, m$ . □

## Graph joining

Let us try some different operation on a graph  $\Gamma$ . Let  $\Gamma_1$  be a graph with an eigenfunction  $f_1$  corresponding to the eigenvalue 1. We take a vertex  $q \in \Gamma$  and a vertex  $p \in \Gamma_1$ . We construct  $\Gamma_0 = \Gamma \cup \Gamma_1$  in such a way that  $\Gamma_0$  contains all the connections of  $\Gamma$  and  $\Gamma_1$  and also in addition we add edges from  $q$  to  $j \in \Gamma_1$  whenever  $j \sim p$ .

**Theorem 2.2.**  $\Gamma_0$  possesses the eigenvalue 1 with eigenfunction  $f_1^{\Gamma_0}$  which coincides with  $f_1$  on  $\Gamma_1$ .

*Proof.* We take,

$$f_1^{\Gamma_0}(i) = \begin{cases} f_1^{\Gamma_1}(i) & \text{if } i \in \Gamma_1 \\ 0 & \text{if } i \in \Gamma \end{cases} \quad (10)$$

We denote

$$e(i) = \sum_{j \in \Gamma_0, j \sim i} f(j) \quad (11)$$

Then we get,  $e(q) = e(p) = 0$

Thus, we have  $\forall i \in \Gamma_0$ ,

$$e(i) = 0$$

Thus  $f_1^{\Gamma_0}$  is a eigenfunction for  $\lambda = 1$ . □

**Example:** Let  $\Gamma$  be a triangle with vertices  $p_1, p_2$  and  $p_3$ . So  $\Gamma$  has the eigenvalues 0 and 1.5 (with  $m_{1.5} = 2$ ). We take  $\Gamma_1$  to be a quadrilateral with vertices  $p_4, p_5, p_6$  and  $p_7$  having no diagonal. Suppose  $p_5$  and  $p_7$  are non adjacent. We join  $p_3$  with  $p_5$  and  $p_7$ . Then the resulting graph  $\Gamma_0$  possesses an eigenvalue 1 with  $m_1 = 2$  having the eigenfunctions,

$$\begin{aligned} f_1^{(1)} &= [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0] \\ f_1^{(2)} &= [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad -1] \end{aligned}$$

**Note:** Suppose  $\Gamma$ ,  $\Gamma_1$  and  $\Gamma_0$  are graphs as described in theorem (2.2). Suppose  $\Gamma_1$  has the eigenvalue 1 with multiplicity  $m_1 = l$  then  $\Gamma_0$  ensures  $\lambda = 1$  with multiplicity at least  $l$ . Here we use the word 'at least' just because, if  $m_1 = l$  in  $\Gamma_1$  then we can have  $m_1 > l$  in  $\Gamma_0$ . It may happen also when  $m_1 = l$  in  $\Gamma$ . For example consider  $\Gamma$  to be a single vertex  $\{q\}$  and  $\Gamma_1 = K_{1,2}$ . We know that,

$$\begin{aligned} m_0 &= 1, \text{ in } \Gamma \text{ and} \\ m_0 &= 1, m_1 = 1, m_2 = 1, \text{ in } \Gamma_1 \end{aligned}$$

Now to get  $\Gamma_0$  we join  $q$  to the vertex  $p \in \Gamma_1$  such that  $n_p = 2$  so that we get  $\Gamma_0 = K_{1,3}$ . So in  $\Gamma_0$ ,  $m_0 = 1$ ,  $m_1 = 2$ ,  $m_2 = 1$ .

### 3 Eigenvalue $\lambda$

#### Motif Doubling :

Let  $\Sigma$  be a motif in  $\Gamma$ . Suppose a real valued function  $f$  satisfies

$$\frac{1}{n_i} \sum_{j \in \Sigma, j \sim i} f(j) = (1 - \lambda)f(i) \text{ for all } i \in \Sigma \text{ and some realvalued } \lambda, \quad (12)$$

where  $n_i$  denote the degree of  $i$  in  $\Gamma$ .

Let  $\Gamma^\Sigma$  is the graph obtained from  $\Gamma$  by doubling  $\Sigma$  in  $\Gamma$  [1]. Then  $\Gamma^\Sigma$  ensures an eigenfunction  $f_\lambda^{\Gamma^\Sigma}$  for the eigenvalue  $\lambda$  which takes the value  $f$  on  $\Sigma$  and  $-f$  on its duplicate and can be defined by,

$$f_\lambda^{\Gamma^\Sigma}(p) = \begin{cases} f(p_\alpha) & \text{if } p = p_\alpha \in \Sigma \\ -f(p_\alpha) & \text{if } p = q_\alpha, \\ 0 & \text{else.} \end{cases} \quad (13)$$

As we have already seen (in [1], [2]) that if we double an edge, a motif of two vertices,  $p_1 p_2$  then

$$\frac{1}{n_{p_1}} f(p_2) = (1 - \lambda)f(p_1), \quad \frac{1}{n_{p_2}} f(p_1) = (1 - \lambda)f(p_2) \quad (14)$$

which admits the solutions

$$\lambda = 1 \pm \frac{1}{\sqrt{n_{p_1} n_{p_2}}}. \quad (15)$$

Here  $\lambda$  is symmetric about 1.

Now we generalise this fact. Suppose  $\Sigma$  consists of the vertices  $p_1, p_2, \dots, p_m$ . Let  $n_{p_i}$  be the degree of  $p_i$  in  $\Gamma$ . Now if we double  $\Sigma$  then from the eigenvalue equation (12) we get,

$$\frac{1}{n_{p_i}} \sum_{p_j \in \Sigma, p_j \sim p_i} f(p_j) = (1 - \lambda)f(p_i) \text{ for all } p_i \in \Sigma \quad (16)$$

which gives,

$$\begin{cases} (\lambda - 1)n_{p_1}f(p_1) + \delta_{p_1 p_2}f(p_2) + \delta_{p_1 p_3}f(p_3) + \dots + \delta_{p_1 p_m}f(p_m) = 0 \\ \delta_{p_2 p_1}f(p_1) + (\lambda - 1)n_{p_2}f(p_2) + \delta_{p_2 p_3}f(p_3) + \dots + \delta_{p_2 p_m}f(p_m) = 0 \\ \delta_{p_3 p_1}f(p_1) + \delta_{p_3 p_2}f(p_2) + (\lambda - 1)n_{p_3}f(p_3) + \dots + \delta_{p_3 p_m}f(p_m) = 0 \\ \vdots \\ \delta_{p_m p_1}f(p_1) + \delta_{p_m p_2}f(p_2) + \delta_{p_m p_3}f(p_3) + \dots + (\lambda - 1)n_{p_m}f(p_m) = 0 \end{cases} \quad (17)$$

where,

$$\delta_{p_i p_j} = \begin{cases} 1 & \text{if } p_i \sim p_j \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

We introduce the matrix  $A = (a_{ij})$ , where

$$a_{ij} = \begin{cases} (\lambda - 1)n_{p_i} & \text{if } i = j \\ \delta_{p_i p_j} & \text{if } i \neq j \end{cases} \quad (19)$$

Then (17) reduces to,

$$AF = O, \text{ where } F = (f(p_1) \ f(p_2) \ \cdots \ f(p_m))^T \text{ and } O = (0 \ 0 \ \cdots \ 0)^T. \quad (20)$$

Thus to get a nonzero solution for  $f$  w.r.t. the systems of equation (17) we have,

$$\det(A) = 0. \quad (21)$$

Now the matrix  $A$  can be written as,

$$A = A_\Sigma + (\lambda - 1)\text{diag}[n_{p_1}, n_{p_2}, \dots, n_{p_m}], \quad (22)$$

where  $A_\Sigma$  is the adjacency matrix of  $\Sigma$ , which is defined by,

$$A_\Sigma = (\delta_{p_i p_j})$$

As a special case, if  $n_{p_1} = n_{p_2} = \cdots = n_{p_m} = k$  then (22) yields,

$$A = A_\Sigma + k(\lambda - 1)I_m. \quad (23)$$

Thus we have the theorem stated as follows,

**Theorem 3.1.** *Let  $\Gamma$  be a graph and  $\Sigma$  be any motif inside  $\Gamma$ . Suppose  $\Gamma^\Sigma$  is obtained by doubling  $\Sigma$  in  $\Gamma$ . Then the additional eigenvalues to the graph  $\Gamma^\Sigma$  are given by  $\det(A) = 0$ , where  $A$  is defined by,*

$$A = (a_{ij}), \text{ such that, } a_{ij} = \begin{cases} (\lambda - 1)n_{p_i} & \text{if } i = j \\ 1 & \text{if } p_i \sim p_j \\ 0 & \text{else.} \end{cases}$$

**An example:** Let  $\Gamma$  be a graph with more than three vertices and  $\Sigma$  consists of the vertices  $p_1, p_2$  and  $p_3$  with degrees  $n_{p_1}, n_{p_2}$ , and  $n_{p_3}$  respectively. Suppose  $p_1 \sim p_2$  and  $p_2 \sim p_3$  and  $p_1, p_3$  are non adjacent. Then from the theorem(3.1) we have,

$$\begin{vmatrix} n_{p_1}(\lambda - 1) & 1 & 0 \\ 1 & n_{p_2}(\lambda - 1) & 1 \\ 0 & 1 & n_{p_3}(\lambda - 1) \end{vmatrix} = 0$$

$$\Rightarrow n_{p_1}n_{p_2}n_{p_3}(\lambda - 1)^3 - (n_{p_1} + n_{p_3})(\lambda - 1) = 0$$

Which gives,

$$\lambda = 1 \text{ or } \lambda = 1 \pm \sqrt{\frac{n_{p_1} + n_{p_3}}{n_{p_1}n_{p_2}n_{p_3}}}$$

And if  $n_{p_1} = n_{p_2} = n_{p_3} = k$ , then the values of  $\lambda$  are given by,

$$\lambda = 1 \text{ or } \lambda = 1 \pm \frac{\sqrt{2}}{k}$$

For very large  $k$ ,  $\lambda \approx 1$ .

### Repeated Duplication of a Motif

As we have already seen in theorem(2.1) that the repeated duplication increases the multiplicity of the eigenvalue 1. Now the question, is does it hold for any  $\lambda$ ? Yes, and we prove it in the following theorem,

**Theorem 3.2.** *Let  $\Sigma$  be a motif in  $\Gamma$ . Let  $\Sigma$  consists of the vertices  $p_1, p_2, \dots, p_m$ . Suppose a realvalued function  $f$  satisfies the equation*

$$\frac{1}{n_{p_i}} \sum_{p_j \in \Sigma, p_j \sim p_i} f(p_j) = (1 - \lambda)f(p_i) \text{ for all } p_i \in \Sigma \text{ and some realvalued } \lambda. \quad (24)$$

Let  $\Gamma^{\Sigma^1}, \Gamma^{\Sigma^2}, \dots, \Gamma^{\Sigma^m}$  be the graph after 1st, 2nd,  $\dots$ ,  $m$ th motif doubling as we described in theorem(2.1). Then  $\Gamma^{\Sigma^i}$ ,  $i = 1, 2, \dots, m$  has the eigenvalue  $\lambda$  with a set of eigenfunctions  $f_j^{\Gamma^{\Sigma^i}}$ ,  $j = 1, 2, \dots, i$  all of which coincides with  $f$  on  $\Sigma$ .

*Proof.* We set,

$$f_j^{\Gamma^{\Sigma^i}}(p) = \begin{cases} f(p_\alpha) & \text{if } p = p_\alpha \in \Sigma \\ f(p_\alpha) & \text{if } p = q_\alpha^{(l)}, j > 1, 1 \leq l \leq j - 1 \\ -j f(p_\alpha) & \text{if } p = q_\alpha^{(j)} \\ 0 & \text{else} \end{cases} \quad (25)$$

for  $i = 1, 2, \dots, m$ .

Then  $f_j^{\Gamma^{\Sigma^i}}$ ,  $j = 1, 2, \dots, i$  are the eigenfunctions corresponding to the eigenvalue  $\lambda$  for the graph  $\Gamma^{\Sigma^i}$ .  $\square$

**An example (star of triangles):** Let  $\Gamma^i$  denote the star with  $i$  triangles *i.e.*  $i$  triangles are joined at a single vertex. Let  $\Gamma$  denote the triangle with vertices  $p_1, p_2$  and  $p_3$ . The eigenvalues of  $\Gamma$  are  $\lambda = 0$  ( $m_0 = 1$ ) and  $\lambda = 1.5$  ( $m_{1.5} = 2$ ). With the eigenfunctions,

$$f_0 = [1 \quad 1 \quad 1]$$

$$f_{1.5}^{(1)} = [0 \quad 1 \quad -1]$$

$$f_{1.5}^{(2)} = [-2 \quad 1 \quad 1]$$

We take  $\Sigma$  consisting of the vertices  $p_2$  and  $p_3$ . Then  $\Gamma^i$  just becomes the  $i-1$  times doubling of  $\Sigma$  in  $\Gamma$  which is  $\Gamma^{\Sigma^{i-1}}$ . So from (14) we have

$$\lambda = 1 \pm \frac{1}{\sqrt{n_{p_1} n_{p_2}}} \\ \Rightarrow \lambda = 0.5, 1.5$$

Thus the eigenvalues of  $\Gamma^i$  are 0, 0.5 and 1.5 with multiplicities 1,  $i-1$  and  $i+1$  respectively.

Now using theorem(3.2) we can get the corresponding eigenfunctions as,

$$f'_0 = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \dots \quad 1 \quad 1]$$

$$f_{0.5}^{\prime(1)} = [0 \quad 1 \quad 1 \quad -1 \quad -1 \quad 0 \quad 0 \quad \dots \quad 0 \quad 0]$$

$$f_{0.5}^{\prime(2)} = [0 \quad 1 \quad 1 \quad 1 \quad 1 \quad -2 \quad -2 \quad \dots \quad 0 \quad 0]$$

$$\vdots$$

$$f_{0.5}^{\prime(i-1)} = [0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \dots \quad 1-i \quad 1-i]$$

$$f_{1.5}^{\prime(1)} = [0 \quad 1 \quad -1 \quad -1 \quad 1 \quad 0 \quad 0 \quad \dots \quad 0 \quad 0]$$

$$f_{1.5}^{\prime(2)} = [0 \quad 1 \quad -1 \quad 1 \quad -1 \quad -2 \quad 2 \quad \dots \quad 0 \quad 0]$$

$$\vdots$$

$$f_{1.5}^{\prime(i-1)} = [0 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad \dots \quad 1-i \quad i-1]$$

$$f_{1.5}^{\prime(i)} = [0 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad \dots \quad 1 \quad -1]$$

$$f_{1.5}^{\prime(i+1)} = [-2 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \dots \quad 1 \quad 1]$$

A different proof of this problem can be found in [3].



## Motif Joining :

Now we do a different operation, motif joining *i.e.* we add a motif  $\Sigma$  to an existing graph  $\Gamma$ . In this section we discuss how can we produce an eigenvalue  $\lambda$  by motif joining.

Let  $\Gamma$  be a graph and  $\Sigma$  be another graph (we call it motif). Let  $\Sigma_c$  be an induced subgraph of  $\Sigma$ . We take a vertex  $p \in \Gamma$  and join it to all  $j \in \Sigma_c$  by an edge and get the resultant graph  $\Gamma^\Sigma$ . Now we have the following theorem associated to this operation,

**Theorem 3.3.** *Suppose there exists a realvalued function  $f$  which satisfies the equation*

$$\frac{1}{n_i} \sum_{j \in \Sigma, j \sim i} f(j) = (1 - \lambda)f(i) \text{ for all } i \in \Sigma \text{ and some realvalue } \lambda \quad (26)$$

and

$$\sum_{j \in \Sigma_c} f(j) = 0, \quad (27)$$

where  $n_i$  is the degree of  $i$  in  $\Gamma^\Sigma$ . Then  $\Gamma^\Sigma$  possesses the eigenvalue  $\lambda$  with an eigenfunction  $f^{\Gamma^\Sigma}$  which coincides with  $f$  on  $\Sigma$ .

*Proof.* We take,

$$f^{\Gamma^\Sigma}(p) = \begin{cases} f(p) & \text{if } p \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

□

**Corollary 3.1.** *Let us join a regular graph  $\Sigma$  to a graph  $\Gamma$  by joining a vertex  $p \in \Gamma$  to all the vertices in  $\Sigma$  and get a new graph  $\Gamma^\Sigma$ . Now if there exists a realvalued function  $f$  which satisfies the equation  $\frac{1}{n_i} \sum_{j \in \Sigma, j \sim i} f(j) = (1 - \lambda)f(i)$  for all  $i \in \Sigma$  and some realvalue  $\lambda$  (where  $n_i$  is the degree of  $i$  in  $\Gamma^\Sigma$ ), then  $\Gamma^\Sigma$  possesses the eigenvalue  $\lambda$  with the eigenfunction  $f^{\Gamma^\Sigma}$  which coincides with  $f$  on  $\Sigma$ .*

*Proof.* Here  $\Sigma_c = \Sigma$ . Since  $f^{\Gamma^\Sigma}$  coincides with  $f$  on  $\Sigma$  and  $\langle f^{\Gamma^\Sigma}, f_0 \rangle = 0$ , where  $f_0$  is the eigenfunction corresponding to the eigenvalues 0, this satisfies the equation(27) □

**Corollary 3.2.** *Let  $\Sigma$  be the complete graph  $K_n$  with  $n$  vertices and  $\Gamma^\Sigma$  be the graph obtained by joining all vertices of  $\Sigma$  to a vertex  $p \in \Gamma$ . Now if there exists a realvalued function  $f$  which satisfies the equation  $\frac{1}{n_i} \sum_{j \in \Sigma, j \sim i} f(j) = (1 - \lambda)f(i)$  for all  $i \in \Sigma$  and some realvalue  $\lambda$  (where  $n_i$  is the degree of  $i$  in  $\Gamma^\Sigma$ ), then  $\Gamma^\Sigma$  possesses the eigenvalue  $\lambda$  with the eigenfunction  $f^{\Gamma^\Sigma}$  which coincides with  $f$  on  $\Sigma$ .*

Moreover the value of the  $\lambda$  is given by  $\lambda = \frac{n+1}{n}$  with  $m_\lambda = n - 1$ .

*Proof.* Since complete graph is regular so the first part of the corollary follows from corollary 3.1.

For the second part we consider the graph  $\Sigma' = \Sigma \cup \{p\}$  obtained by taking all the edges of  $\Sigma$  and edges  $(i, p)$  for all  $i \in \Sigma$ . Then  $\Sigma'$  becomes  $K_{n+1}$ . It has the eigenvalue  $\lambda = \frac{n+1}{n}$  with  $n-1$  eigenfunctions  $f_\lambda^i$ ,  $i = 1, 2, \dots, n-1$  which takes the value 0 at  $p$ , and  $\sum_{j \in \Sigma} f_\lambda^i(j) = 0$  for  $i = 1, 2, \dots, n-1$ . Since  $f_\lambda^i(p) = 0$  and  $\sum_{j \in \Sigma} f_\lambda^i(j) = 0$ ,  $f_\lambda^i$  can be extended to an eigenfunction of  $\Gamma^\Sigma$  for all  $i = 1, 2, \dots, n-1$ . Thus we can have  $\lambda = \frac{n+1}{n}$  for the graph  $\Gamma^\Sigma$  with the multiplicity  $m_\lambda = n-1$ .  $\square$

**As an example:** If we consider  $\Gamma$  and  $\Sigma$  both to be  $K_2$  then by this corollary we get  $\lambda = 1.5$  with an eigenfunction

$$f_{1.5} = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}.$$

**Corollary 3.3.** Suppose we join  $\Sigma_c$  to the vertices  $p_1, p_2, \dots, p_k \in \Gamma$  instead of joining it to a single vertex  $p$  unlike in corollary(3.1). Now if there exists a realvalued function  $f$  which satisfies the equation  $\frac{1}{n_i} \sum_{j \in \Sigma, j \sim i} f(j) = (1 - \lambda)f(i)$  for all  $i \in \Sigma$  and some real value  $\lambda$  (where  $n_i$  is the degree of  $i$  in  $\Gamma^\Sigma$ ) and  $\sum_{j \in \Sigma_c} f(j) = 0$ . Then also the theorem(3.3) holds trivially.

*Proof.* We take,

$$f^{\Gamma^\Sigma}(p) = \begin{cases} f(p) & \text{if } p \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

Hence  $f^{\Gamma^\Sigma}$  becomes an eigenfunction corresponding to the eigenvalue  $\lambda$  which coincides with  $f$  on  $\Sigma$ .  $\square$

**Corollary 3.4.** Now we wanted to combine to combine corollary(3.2) and corollary(3.3). That is, we take  $\Sigma$  to be  $K_n$  and we obtain  $\Gamma^\Sigma$  by joining  $\Sigma$  to the vertices  $p_1, p_2, \dots, p_k \in \Gamma$ . Now suppose for a realvalued function  $f$  the equation  $\frac{1}{n_i} \sum_{j \in \Sigma, j \sim i} f(j) = (1 - \lambda)f(i)$  holds for all  $i \in \Sigma$  and some realvalue  $\lambda$  (where  $n_i$  is the degree of  $i$  in  $\Gamma^\Sigma$ ). Here we get the eigenvalue  $\lambda = \frac{n+k}{n+k-1}$  with the same multiplicity  $m_\lambda = n-1$ .

*Proof.* This fact can be proved in a similar way as we did in corollary(3.2).

For this we Consider the graph  $\Sigma' = \Sigma \cup \{p_1, p_2, \dots, p_k\}$  obtained by taking all the edges of  $\Sigma$  and the edges  $(i, p_j)$  for all  $i \in \Sigma$  and  $j = 1, 2, \dots, k$ .

Now if we join each  $p_s$  to  $p_t$  for  $1 \leq s < t \leq k$ , then  $\Sigma'$  becomes  $K_{n+k}$ , which has the eigenvalue  $\lambda = \frac{n+k}{n+k-1}$  with  $n-1$  eigenfunctions  $f_\lambda^i$ ,  $i = 1, 2, \dots, n-1$  that takes the value 0 at  $p_1, p_2, \dots, p_k$ , and  $\sum_{j \in \Sigma} f_\lambda^i(j) = 0$  for  $i = 1, 2, \dots, n-1$ .

Note that if we delete an edge  $(p_l, p_j)$ ,  $\lambda$  becomes unchanged, since  $f_\lambda^i(p_l) = f_\lambda^i(p_j) = 0$  for all  $l, j = 1, 2, \dots, k$  and for all  $i = 1, 2, \dots, n-1$ .

Thus we have the eigenvalue  $\lambda = \frac{n+k}{n+k-1}$  with the multiplicity  $m_\lambda = n-1$  for the graph  $\Sigma'$  with the eigenfunctions  $f_\lambda^i$  for  $i = 1, 2, \dots, n-1$ .

Since  $f_\lambda^i(p_j) = 0$  for  $j = 1, 2, \dots, k$  and  $\sum_{j \in \Sigma} f_\lambda^i(j) = 0$ ,  $f_\lambda^i$  can be extended to an eigenfunction of  $\Gamma^\Sigma$  for  $i = 1, 2, \dots, n-1$ . Thus we have the eigenvalue  $\lambda = \frac{n+k}{n+k-1}$  for the graph  $\Gamma^\Sigma$  with the multiplicity  $m_\lambda = n-1$ .  $\square$

We can also generalize the fact in the theorem(3.3) as follows: Let  $\Gamma$  be a graph and  $\Sigma$  be another graph. Let  $\Sigma_{c_1}, \Sigma_{c_2}, \dots, \Sigma_{c_k}$  be induced subgraphs (not necessarily having disjoint vertex sets) of  $\Sigma$ . We take distinct vertices  $p_1, p_2, \dots, p_k \in \Gamma$ . We join all vertices  $j \in \Sigma_{c_l}$  by an edge to  $p_l$  for all  $l = 1, 2, \dots, k$  and produce  $\Gamma^\Sigma$ . Now we get the generalized result which is stated as follows:

**Theorem 3.4.** *Suppose there exists a realvalued function  $f$  which satisfies*

$$\frac{1}{n_i} \sum_{j \in \Sigma, j \sim i} f(j) = (1 - \lambda)f(i) \text{ for all } i \in \Sigma \text{ and some realvalue } \lambda \quad (28)$$

and

$$\sum_{j \in \Sigma_{c_l}} f(j) = 0 \text{ for } l = 1, 2, \dots, k, \quad (29)$$

where  $n_i$  is the degree of  $i$  in  $\Gamma^\Sigma$ . Then  $\Gamma^\Sigma$  possesses the eigenvalue  $\lambda$  with an eigenfunction  $f^{\Gamma^\Sigma}$  which coincides with  $f$  on  $\Sigma$ .

*Proof.* We take,

$$f^{\Gamma^\Sigma}(p) = \begin{cases} f(p) & \text{if } p \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

Then  $f^{\Gamma^\Sigma}$  is the required eigenfunction for the graph  $\Gamma^\Sigma$ .  $\square$

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